

NILPOTENT n -TUPLES IN $SU(2)$

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ABSTRACT: Let F_n/Γ_n^q denote the finitely generated free q -nilpotent group. We describe the connected components of the spaces of homomorphisms $\text{Hom}(F_n/\Gamma_n^q, SU(2))$, $\text{Hom}(F_n/\Gamma_n^q, SO(3))$ and $\text{Hom}(F_n/\Gamma_n^q, U(2))$. We also describe the homotopy type of the classifying space $B(q, G)$ of transitionally q -nilpotent G -bundles, for $G = Q_{2^q}$ and $SU(2)$, where Q_{2^q} is the generalized quaternion group of order 2^q .

1 INTRODUCTION

Spaces of homomorphisms have been of interest for a long time now. One of the basic problems is to compute the number of its connected components. In this paper we mainly focus on the spaces of homomorphisms of the finitely generated free q -nilpotent groups denoted by F_n/Γ_n^q on the Lie group $SU(2)$. As noted in [7] the case $q = 3$ is the same as computing the space of almost commuting tuples of $SU(2)$ which were completely described in [2]. We generalize this for q -nilpotent tuples with $q \geq 4$. We prove that all non-abelian nilpotent subgroups of $SU(2)$ are conjugated to the quaternion group Q_8 or to one of the generalized quaternions Q_{2^q} of order 2^q . Using this we prove the following.

Theorem 1.1. *Let $q \geq 3$ and $n \geq 2$. Then*

$$\text{Hom}(F_n/\Gamma_n^q, SU(2)) \cong \text{Hom}(\mathbb{Z}^n, SU(2)) \sqcup \bigsqcup_{C(n,q)} PU(2)$$

where

$$C(n, q) = \frac{2^{n-2}(2^n - 1)(2^{n-1} - 1)}{3} + (2^n - 1)(2^{(q-3)(n-1)} - 1)2^{2n-3}.$$

Two consequences of this result are the descriptions of the q -nilpotent tuples for $SO(3)$ and $U(2)$. The double covering map $\pi: SU(2) \rightarrow SO(3)$ induces a surjective map

$$\text{Hom}(F_n/\Gamma_n^q, SU(2)) \rightarrow \text{Hom}(F_n/\Gamma_n^{q-1}, SO(3)),$$

which we can use to compute the connected components of $\text{Hom}(F_n/\Gamma_n^q, SO(3))$. The case $q = 2$ was originally proved in [6].

Corollary 1.2. *Let $q \geq 3$, $n \geq 2$ and $\text{Hom}(\mathbb{Z}^n, SO(3))_1$ be the connected component that contains the identity n -tuple (I, \dots, I) . Then*

$$\text{Hom}(F_n/\Gamma_n^q, SO(3)) \cong \text{Hom}(\mathbb{Z}^n, SO(3))_1 \sqcup \bigsqcup_{M(n,2)} S^3/Q_8 \sqcup \bigsqcup_{M(n,q)} S^3/C_4,$$

where $M(n, 2) = \frac{(2^n-1)(2^{n-1}-1)}{3}$, $M(n, q) = (2^n - 1)(2^{(q-3)(n-1)} - 1)2^{n-2}$ and C_4 is a cyclic group of order 4.

The group homomorphism $S^1 \times SU(2) \rightarrow U(2)$ given by $(\lambda, X) \mapsto \lambda X$ induces a surjective map

$$(S^1)^n \times \text{Hom}(F_n/\Gamma_n^q, SU(2)) \rightarrow \text{Hom}(F_n/\Gamma_n^q, U(2))$$

and modding out the left hand side by the natural diagonal action of $(\mathbb{Z}/2)^n$ we get a homeomorphism.

Corollary 1.3. *Let $q \geq 3$ and $n \geq 2$, then*

$$\text{Hom}(F_n/\Gamma_n^q, U(2)) \cong \text{Hom}(\mathbb{Z}^n, U(2)) \sqcup \bigsqcup_{\widetilde{M}(n,3)} (S^1)^n \times_{(\mathbb{Z}/2)^2} PU(2) \sqcup \bigsqcup_{\widetilde{M}(n,q)} (S^1)^n \times_{(\mathbb{Z}/2)} PU(2),$$

where $\widetilde{M}(n, q) = M(n, q-1)$ is as in the previous Corollary.

Let $m > 2$. We finish up section 2 with a description of the space of representations $\text{Rep}(F_n/\Gamma_n^3, U(m))$ (the orbit space associated to the conjugation action of $U(m)$ over $\text{Hom}(F_n/\Gamma_n^3, U(m))$) that expresses it as a union of almost commuting tuples of block matrix subgroups of $U(m)$ that can be identified with the direct product $U(m_1) \times \cdots \times U(m_k)$ where $m_1 + \cdots + m_k = m$.

In section 3, using the homotopy stable decomposition obtained in [7] for the spaces $\text{Hom}(F_n/\Gamma_n^q, G)$ (for G a real algebraic linear group), we compute the stable homotopy type of the spaces $\text{Hom}(F_n/\Gamma_n^q, SU(2))$ and $\text{Hom}(F_n/\Gamma_n^q, SO(3))$ after one suspension.

The spaces $\text{Hom}(F_n/\Gamma_n^q, G)$ assemble into a simplicial space and we denote its geometric realization as $B(q, G)$. These spaces were originally introduced in [1]. In section 4 we describe the \mathbb{F}_2 -cohomology rings $H^*(B_{\text{com}}Q_{2^q}; \mathbb{F}_2)$ and $H^*(B(3, Q_{16}); \mathbb{F}_2)$ where $B_{\text{com}}G = B(2, G)$. Also, we compute the homotopy type of the classifying spaces $B(q, SU(2))$ as the homotopy pushouts

$$\begin{array}{ccc} PU(2) \times_{N_q} B(q-1, Q_{2^q}) & \longrightarrow & B(q-1, SU(2)) \\ \downarrow & & \downarrow \\ PU(2) \times_{N_q} BQ_{2^q} & \longrightarrow & B(q, SU(2)), \end{array}$$

where N_q is the normalizer of the dihedral group $D_{2^{q-1}}$ in $PU(2)$.

2 NILPOTENT TUPLES IN $SU(2)$

Let $T \subset SU(2)$ denote the maximal torus

$$T = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix} \mid \lambda \in \mathbb{C} \text{ and } |\lambda| = 1 \right\}$$

and let $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. A straightforward calculation shows:

Lemma 2.1. *Let $x, y \in T$. Then $wx = \bar{x}w$ and*

- $[x, y] = 1$;
- $[x, wy] = x^2$;

- $[wx, y] = \bar{y}^2$;
- $[wx, wy] = x^2 \bar{y}^2$.

Definition 2.2. Let Q be a group, define inductively $\Gamma^1(Q) = Q$; $\Gamma^{q+1}(Q) = [\Gamma^q(Q), Q]$. This is called the descending central series of Q

$$\cdots \subset \Gamma^q(Q) \subset \cdots \subset \Gamma^2(Q) \subset Q.$$

A group Q is *nilpotent* if $\Gamma^{q+1}(Q) = 1$ for some q . The least such integer q is called the *nilpotency class* of Q .

Let $\xi_n = \begin{pmatrix} e^{2\pi i/n} & 0 \\ 0 & e^{-2\pi i/n} \end{pmatrix} \in T$ be an n -th root of unity and let μ_n stand for the subgroup generated by ξ_n . The general quaternions

$$Q_{2q+1} := \mu_{2q} \cup w\mu_{2q}$$

are of nilpotency class q for any $q > 1$. This follows from the fact that $[\xi_{2^n}, w] = \xi_{2^n}^2 = \xi_{2^{n-1}}$ for all $n \geq 1$.

Lemma 2.3. *Let $H \subset T \cup wT$ be a subgroup. Suppose $\Gamma^r(H) = \mu_{2^n}$ for some $n > 0$. Then $\Gamma^{r-1}(H) = \mu_{2^{n+1}}$ for all $r > 2$, and for $r = 2$ there exists $t \in T$ such that $tH\bar{t} = Q_{2^{n+2}}$.*

Proof. We prove the case $r = 2$. Suppose $\Gamma^2(H) = [H, H] = \mu_{2^n}$. Since the commutator generates μ_{2^n} , there exist $z_0, wx_0 \in H$, with $x_0 \in T$ such that $[z_0, wx_0] = \xi_{2^n}$. Let $z \in H \cap T$. Then $[z, wx_0] = z^2$ is a power of ξ_{2^n} , that is, z is a power of $\pm \xi_{2^{n+1}}$ and $z \in \mu_{2^{n+1}}$. Let $wy \in H \cap wT$, where $y \in T$. The commutator $[wy, wx_0] = y^2 \overline{x_0}^2 = \xi_{2^n}^p$ for some $p > 0$, therefore $y = \pm \xi_{2^{n+1}}^p x_0$. Therefore $H \subseteq \mu_{2^{n+1}} \cup w\mu_{2^{n+1}}x_0$. To get the equality, consider the element $z_0 \in H$. We have 2 possibilities:

- z_0 is diagonal, and $z_0^2 = \xi_{2^n}$ implies $z_0 = \pm \xi_{2^{n+1}}$.
- z_0 is anti-diagonal, so that $z_0 = wz'_0$, with $z'_0 = \pm \xi_{2^{n+1}}x_0$.

In both cases z_0 and wx_0z_0 generate $\mu_{2^{n+1}} \cup w\mu_{2^{n+1}}x_0$. Conjugating any element $w\xi_{2^{n+1}}^p x_0$ by $t = \sqrt{x_0} \in T$ we obtain

$$\sqrt{x_0} w \xi_{2^{n+1}}^p x_0 \overline{\sqrt{x_0}} = w \overline{\sqrt{x_0}} \xi_{2^{n+1}}^p x_0 \overline{\sqrt{x_0}} = w \xi_{2^{n+1}}^p.$$

This is independent of the choice of the branch cut for $\sqrt{x_0}$. Hence $\Gamma^1(tH\bar{t}) = tH\bar{t} \subseteq \mu_{2^{n+1}} \cup w\mu_{2^{n+1}}$.

For $r > 2$ the same arguments without the anti-diagonal matrices cases prove the result, since $\Gamma^{r-1}(H) \subset T$. \square

Lemma 2.4. *Let $X, Y \in SU(2)$.*

1. *If $[X, Y] = I$ and $g \in SU(2)$ diagonalizes X , then it also diagonalizes Y .*
2. *If $[X, Y] = -I$ and $g \in SU(2)$ diagonalizes X , then $gXg^{-1} = \pm \xi_4$ and $gYg^{-1} \in wT$.*

Proof. 2. Let g be a matrix that conjugates X to a diagonal matrix with eigenvalues $\lambda, \bar{\lambda}$. Let $X' = gXg^{-1}$ and $Y' = gYg^{-1}$. Choose a non-zero vector $v \in E_\lambda$, the eigenspace of X associated to λ . Since $[X', Y'] = -I$ we get

$$X'Y'v = -Y'X'v = -\lambda Y'v.$$

Hence $-\lambda$ is an eigenvalue of X so that $-\lambda = \bar{\lambda}$, which implies $\lambda = \pm i \in \mathbb{C}$. Therefore $X' = \pm \xi_4$. Now, $v \in E_i$ and $Y'v \in E_{-i}$ tells us that Y is conjugated by g into an anti-diagonal matrix. \square

Proposition 2.5. *Let $H \subset SU(2)$ be a nilpotent subgroup. Then either H is abelian or there exists a unique $r \geq 2$ and an element $g \in SU(2)$ such that*

$$gHg^{-1} = Q_{2^{r+1}}.$$

Proof. Suppose H is non-abelian. Then, there exists a unique $r > 1$ such that $\Gamma^{r+1}(H) = I$ and $\Gamma^r(H) \neq I$. This implies that $\Gamma^r(H)$ sits inside the center of H . Non-abelian subgroups of $SU(2)$ have center contained in $\{\pm I\}$ and therefore $\Gamma^r(H) = [H, \Gamma^{r-1}(H)] = \mu_2$. Fix an element X in $\Gamma^{r-1}(H)$. For any $h \in H$, the commutator $[X, h]$ is inside $\{\pm I\}$. By Lemma 2.4, H can be conjugated to $T \cup wT$ by an element $g \in SU(2)$ that diagonalizes X . Now, $\Gamma^r(gHg^{-1}) = \mu_2$ and applying Lemma 2.3 inductively we get that there exists $t \in T$, such that $tgH(gt)^{-1} = Q_{2^{r+1}}$. \square

Lemma 2.6. *Let $x, y \in T$. Then*

- $xyx^{-1} = y$;
- $(wx)y(wx)^{-1} = \bar{y}$;
- $x(wy)x^{-1} = x^2(wy) = x^2\bar{y}w = w\bar{x}^2y$;
- $(wx)(wy)(wx)^{-1} = x^2\bar{y}^2(wy) = x^2\bar{y}^3w = w\bar{x}^2y^3$.

Lemma 2.7. 1. *Let $q \geq 3$. The abelian subgroups of Q_{2^q} are all subgroups of $\mu_{2^{q-1}}$ or $\{\pm I, \pm wx\}$ where x is an element of $\mu_{2^{q-1}}$.*

2. *Let $q \geq 4$. The non-abelian subgroups of Q_{2^q} are of the form $\mu_{2^{r-1}} \cup wx\mu_{2^{r-1}}$ where $x = (\xi_{2^{q-1}})^p$ with $0 \leq p < 2^{q-r}$ and $3 \leq r \leq q$.*

Proof. 1. By Lemma 2.1, any subgroup containing an element of $\mu_{2^{q-1}} - \{\pm I\}$ and one of $w\mu_{2^{q-1}}$ has non-trivial commutator.

2. By the proof of Lemma 2.3 any non-abelian nilpotent subgroup of $T \cup wT$ has the form $\mu_{2^n} \cup wx\mu_{2^n}$ for some $x \in T$. Therefore the non-abelian subgroups of Q_{2^q} are $\mu_{2^{r-1}} \cup wx\mu_{2^{r-1}}$ where x is not in $\mu_{2^{r-1}}$. \square

Lemma 2.8. *The normalizer of Q_{2^q} in $SU(2)$ is:*

1. *The Binary Octahedral group which is generated by $\left\{ \xi_8, w, \frac{1}{2} \begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix} \right\}$ for $q = 3$;*
2. *$Q_{2^{q+1}}$ for $q > 3$.*

Proof. For this it is more convenient to think of $SU(2)$ as the group of unit quaternions, which enables one to think geometrically in terms of rotations in the space of purely imaginary quaternions, a space we shall identify with \mathbb{R}^3 . If $q = \cos(\theta) + \sin(\theta)\hat{q}$ where \hat{q} is a unit length imaginary quaternion, then $v \mapsto qvq^{-1}$ maps imaginary quaternions to imaginary quaternions and is a rotation around \hat{q} of angle 2θ . An arbitrary quaternion can be written as $a + v$ where a is the real part and v is purely imaginary. We have $q(a + w)q^{-1} = a + qvq^{-1}$, so that conjugation by q preserves the real part and rotates the imaginary part. This rule that associates a rotation of \mathbb{R}^3 to each unit quaternion is the double cover homomorphism $SU(2) \rightarrow SO(3)$.

The normalizer of Q_{2^q} consists of those unit quaternions whose corresponding rotations are the orientation preserving symmetries of the set of imaginary parts of the elements of Q_{2^q} . As quaternions, the elements of Q_{2^q} are $\cos(2\pi k/2^{q-1}) + i \sin(2\pi k/2^{q-1})$ and $j \cos(2\pi k/2^{q-1}) + k \sin(2\pi k/2^{q-1})$ for $k = 0, 1, \dots, 2^{q-1} - 1$. The imaginary parts are then 2^{q-2} points on the i -axis together with the vertices of a regular 2^{q-1} -gon in the jk -plane. For $q > 3$, any orientation preserving isometry of \mathbb{R}^3 preserving this set of points must separately preserve the 2^{q-1} -gon and the points on the i -axis, and thus must be a rotation of \mathbb{R}^3 whose restriction to the plane of the 2^{q-1} -gon is a symmetry of the 2^{q-1} -gon. Each of the 2^q elements of the dihedral group of symmetries of the 2^{q-1} -gon extends to a unique rotation of \mathbb{R}^3 (the rotations in the dihedral group extend to rotations around the i -axis, while the reflections extend to rotations of angle π around the axis of the reflection), and each such rotation of \mathbb{R}^3 has two quaternions inducing it; these 2^{q+1} quaternions are easily seen to be the elements of $Q_{2^{q+1}}$.

Now, if $q = 3$, there is extra symmetry because the 2^{q-1} -gon in the jk -plane is just a square, and the $2^{q-2} = 2$ points on the i -axis together with that square form a regular octahedron. Therefore the normalizer of Q_{2^3} is the preimage in $SU(2)$ of the groups of orientation preserving symmetries of the octahedron in $SO(3)$; this preimage is known as the Binary Octahedral group. The generator $\frac{1}{2} \begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix}$ listed in the statement of the proposition (which is the only generator not in Q_{2^4}), corresponds to a rotation of $2\pi/3$ around a line connecting the centers of two opposite faces of the octahedron. This extra symmetry does *not* separately preserve the i -axis and the square $\{\pm j, \pm k\}$. \square

Remark 2.9. Here's an alternative "matrix-based" proof of the second part. Let $g \in N_{SU(2)}(Q_{2^q})$. Suppose $g\xi_4 g^{-1} = wx$ for some $x \in \mu_{2^{q-1}}$. Then $g\xi_8 g^{-1}$ is an element of order 8, and must lie in some subgroup of $\mu_{2^{q-1}}$, which is a contradiction. Therefore $g\xi_4 g^{-1}$ is a diagonal matrix and this can only be $\pm\xi_4$. By Lemma 2.4, g is in $T \cup wT$. Using Lemma 2.6, $\xi_{2^q} \in N_{SU(2)}(Q_{2^q})$ and for any $g \in N_{SU(2)}(Q_{2^q})$, g^2 is in Q_{2^q} , and thus $N_{SU(2)}(Q_{2^q}) = Q_{2^{q+1}}$.

Theorem 2.10. *Let $q \geq 3$ and $n \geq 2$. Then*

$$\mathrm{Hom}(F_n/\Gamma_n^q, SU(2)) \cong \mathrm{Hom}(\mathbb{Z}^n, SU(2)) \sqcup \bigsqcup_{C(n,q)} PU(2)$$

where

$$C(n, q) = \frac{2^{n-2}(2^n - 1)(2^{n-1} - 1)}{3} + (2^n - 1)(2^{(q-3)(n-1)} - 1)2^{2n-3}.$$

Proof. Consider the map $SU(2) \times (Q_{2^q})^n \rightarrow \text{Hom}(F_n/\Gamma_n^q, SU(2))$ given as conjugation by elements of $SU(2)$. By condition 2 of Proposition 2.5 this map is surjective when we restrict to non-commuting tuples. Modding out by the center $Z(SU(2)) = \{\pm I\}$ we have the induced surjective map

$$\psi: PU(2) \times [(Q_{2^q})^n - \text{Hom}(\mathbb{Z}^n, Q_{2^q})] \rightarrow \text{Hom}(F_n/\Gamma_n^q, SU(2)) - \text{Hom}(\mathbb{Z}^n, SU(2)).$$

Fix an element $x \in (Q_{2^q})^n - \text{Hom}(\mathbb{Z}^n, Q_{2^q})$ and let $g, h \in SU(2)$. Then $gxg^{-1} = h x h^{-1}$ implies that gh^{-1} commutes with a diagonal matrix and with an anti-diagonal matrix. Thus $gh^{-1} = \pm I$. This shows that the map ψ is injective restricted to each connected component. Now, let \sim be the equivalence relation on $(Q_{2^q})^n - \text{Hom}(\mathbb{Z}^n, Q_{2^q})$ defined by: two elements are equivalent if they are conjugated to one another by some element in $PU(2)$. We get the homeomorphism

$$PU(2) \times [(Q_{2^q})^n - \text{Hom}(\mathbb{Z}^n, Q_{2^q})] / \sim \cong \text{Hom}(F_n/\Gamma_n^q, SU(2)) - \text{Hom}(\mathbb{Z}^n, SU(2)).$$

Let $\text{Gen}(n, Q_{2^r}) \subset (Q_{2^r})^n$ be the subset of n -tuples that generate Q_{2^r} . The normalizer $N_{SU(2)}(Q_{2^r})$ acts on $\text{Gen}(n, Q_{2^r})$ by conjugation. The inclusions $\text{Gen}(n, Q_{2^r}) \subset (Q_{2^r})^n$ induce a bijective function

$$\bigsqcup_{r=3}^q \text{Gen}(n, Q_{2^r}) / N_{SU(2)}(Q_{2^r}) \rightarrow [(Q_{2^q})^n - \text{Hom}(\mathbb{Z}^n, Q_{2^q})] / \sim.$$

The action of $N_{SU(2)}(Q_{2^r})$ on $\text{Gen}(n, Q_{2^r})$ is free once we take quotient by the center of the group. Thus

$$|[(Q_{2^q})^n - \text{Hom}(\mathbb{Z}^n, Q_{2^q})] / \sim| = \sum_{r=3}^q \frac{|\text{Gen}(n, Q_{2^r})|}{|N_{PU(2)}(D_{2^{r-1}})|}.$$

By Lemma 2.8 we know the order of $N_{PU(2)}(D_{2^{r-1}})$. It remains to count the number of elements in $\text{Gen}(n, Q_{2^r})$, for which we use an inclusion–exclusion argument: a tuple generates Q_{2^r} if and only if its elements don't all come from a single proper maximal subgroup of Q_{2^r} . So if M_1, M_2, \dots, M_m are the maximal subgroups of Q_{2^r} , we have

$$|\text{Gen}(n, Q_{2^r})| = |Q_{2^r}|^n - \sum_i |M_i|^n + \sum_{i,j} |M_i \cap M_j|^n - \sum_{i,j,k} |M_i \cap M_j \cap M_k|^n + \dots$$

For Q_8 , there are three maximal subgroups, all isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$. the intersection of any two of them is the center of Q_8 , $\pm I$. So we get

$$|\text{Gen}(n, Q_8)| = 8^n - 3 \cdot 4^n + 3 \cdot 2^n - 2^n = 2^{n+1}(2^n - 1)(2^{n-1} - 1).$$

For $r \geq 4$, it follows from 2.7 that things are pretty much the same: there are just three maximal subgroups of Q_{2^r} , namely, $\mu_{2^{r-1}}$, $\mu_{2^{r-2}} \cup w\mu_{2^{r-2}}$ and $\mu_{2^{r-2}} \cup w\xi_{2^{2-1}}\mu_{2^{r-2}}$. And the intersection of any pair of them is $\mu_{2^{r-2}}$, so we get

$$|\text{Gen}(n, Q_{2^r})| = (2^r)^n - 3 \cdot (2^{r-1})^n + 3 \cdot (2^{r-2})^n - (2^{r-2})^n = 2^{(r-2)n+1}(2^n - 1)(2^{n-1} - 1).$$

□

Two immediate applications of the previous Theorem are for nilpotent tuples in $SO(3)$ and $U(2)$. We study first the ones in $SO(3) \cong PU(2)$. Consider $\pi: SU(2) \rightarrow PU(2)$ the quotient homomorphism. Let H be a nilpotent subgroup of $PU(2)$ of nilpotency class m . We have that $\Gamma^m(\pi^{-1}(H)) = \{\pm I\}$ and thus $\pi^{-1}(H)$ has nilpotency class $m + 1$. Similarly, the image of any group of nilpotency class $m > 1$ in $SU(2)$ will have nilpotency class $m - 1$ in $PU(2)$. Therefore, for any $q \geq 3$ we have a surjective map

$$\pi_*: \text{Hom}(F_n/\Gamma_n^q, SU(2)) \rightarrow \text{Hom}(F_n/\Gamma_n^{q-1}, PU(2)).$$

The restriction $\pi_*^{-1}(C) \rightarrow C$ to any connected component C of $\text{Hom}(F_n/\Gamma_n^{q-1}, PU(2))$, is a 2^n -fold covering map (see [5, Lemma 2.2]). From Theorem 2.10 we know that the connected components of $\text{Hom}(F_n/\Gamma_n^q, SU(2))$ are either $\text{Hom}(\mathbb{Z}^n, SU(2))$ or K_i , components homeomorphic to $PU(2)$. Each of the later components consist of the representations that are conjugated (by elements of $PU(2)$) to a fixed surjective homomorphism $F_n/\Gamma_n^q \rightarrow Q_{2^r}$ with $3 \leq r \leq q$. The image under π_* of these components are homomorphisms conjugated to the corresponding epimorphism $F_n/\Gamma_n^{q-1} \rightarrow D_{2^{r-1}}$. We have two different cases. First, if $r \geq 4$, then $\pi_*(K_i) \cong PU(2)/Z(D_{2^{r-1}})$ and recall that $Z(D_{2^{r-1}}) = \langle \pi(\xi_4) \rangle$ has order 2. Hence the restriction $\pi_*|_{K_i}: K_i \rightarrow \pi_*(K_i)$ is a 2-fold covering map. When $r = 3$, $Z(D_4) = D_4$, and $\pi|_{K_i}: K_i \rightarrow \pi_*(K_i) \cong PU(2)/D_4$ is a 4-fold covering map. With these covering maps, we can easily count the connected components of $\text{Hom}(F_n/\Gamma_n^q, SO(3))$. The case $q = 2$ was first computed in [6].

Corollary 2.11. *Let $q \geq 3$, $n \geq 2$ and $\text{Hom}(\mathbb{Z}^n, SO(3))_1$ be the connected component that contains the identity n -tuple (I, \dots, I) . Then*

$$\text{Hom}(F_n/\Gamma_n^q, SO(3)) \cong \text{Hom}(\mathbb{Z}^n, SO(3))_1 \sqcup \bigsqcup_{M(n,2)} S^3/Q_8 \sqcup \bigsqcup_{M(n,q)} S^3/C_4,$$

where $M(n, 2) = \frac{(2^n-1)(2^{n-1}-1)}{3}$, $M(n, q) = (2^n - 1)(2^{(q-3)(n-1)} - 1)2^{n-2}$ and C_4 is the cyclic group of order 4 generated by ξ_4 .

Now we discuss the situation for $U(2)$. Any matrix in $X \in U(2)$ can be written as $\sqrt{\det(X)}X'$, where $X' \in SU(2)$. In this decomposition, $[X, Y] = [X', Y']$ for any X, Y in $U(2)$. Consider the map

$$(S^1)^n \times \text{Hom}(F_n/\Gamma_n^q, SU(2)) \rightarrow \text{Hom}(F_n/\Gamma_n^q, U(2))$$

given by $(\lambda_1, \dots, \lambda_n, x_1, \dots, x_n) \mapsto (\lambda_1 x_1, \dots, \lambda_n x_n)$. By the previous observation this is a surjective map. These representations are uniquely determined up to a negative sign, that is, we have a homeomorphism

$$(S^1)^n \times_{(\mathbb{Z}/2)^n} \text{Hom}(F_n/\Gamma_n^q, SU(2)) \cong \text{Hom}(F_n/\Gamma_n^q, U(2)).$$

Using Theorem 2.10 we get that the connected components of this space are homeomorphic to $(S^1)^n \times_{(\mathbb{Z}/2)^n} \text{Hom}(\mathbb{Z}^n, SU(2)) \cong \text{Hom}(\mathbb{Z}^n, U(2))$ or $(S^1)^n \times_H PU(2)$ where H is a subgroup of $(\mathbb{Z}/2)^n$. The first component corresponds to the commuting n -tuples in $U(2)$ and the later to the r -nilpotent tuples with $2 \leq r \leq q$. We can see the $(\mathbb{Z}/2)^n$ action as an action on

the set of indexes of the connected components, which can be represented by elements of $\text{Hom}(F_n/\Gamma_n^q, Q_{2^q})$. To count the number of connected components, let $\vec{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)$ with each $\varepsilon_i = \pm 1$ be an arbitrary element in $(\mathbb{Z}/2)^n$ and $\vec{x} = (x_1, \dots, x_n)$ a non-commutative n -tuple in $\text{Hom}(F_n/\Gamma_n^q, Q_{2^q})$. Then the stabilizer of this element consists of

$$\text{Stab}(\vec{x}) = \{(\varepsilon_1, \dots, \varepsilon_n) \mid (\varepsilon_1 x_1, \dots, \varepsilon_n x_n) = (g x_1 g^{-1}, \dots, g x_n g^{-1})\}$$

for some $g \in SU(2)$. That is, such g either commutes or anticommutes with all x_i . We have several cases. Let $\vec{\varepsilon} \in \text{Stab}(\vec{x})$ be a non trivial element.

Case 1: Suppose some x_i lies in the torus T .

- If $\varepsilon_i = 1$, $x_i = g x_i g^{-1}$ and thus g must also lie in T . To generate an r -nilpotent subgroup of Q_{2^q} with x_i , we need at least one more element of the form $\xi_{2^q}^k w$ for some k . This element does not commute with any element of $T - \{\pm I\}$ and only anticommutes with $\pm \xi_4$. Thus, the only choices for g are $\pm I$ and $\pm \xi_4$.

- If $\varepsilon_i = -1$, $-x_i = g x_i g^{-1}$ and by Lemma 2.4, $x_i = \pm \xi_4$ and $g \in Tw$. Again, in the n -tuple there must be an element of the form $x_j = \xi_{2^q}^k w$ for some k . The only elements in Tw that commute or anticommute with x_j are $g = \pm \xi_{2^q}^k w$ or $g = \pm \xi_4 \xi_{2^q}^k w$. Note that in this case, the remaining elements in the n -tuple can only be of the form $\pm I, \pm \xi_4, \pm \xi_{2^q}^k w$ or $\pm \xi_4 \xi_{2^q}^k w$, which generates a copy of Q_8 .

Case 2: Suppose all x_i lie in Tw . Let $x_i = \xi_{2^q}^k w$.

- Suppose \vec{x} is an r -nilpotent tuple with $r \geq 3$. Then there is at least one $x_j = \xi_{2^q}^l w$ that is different from $\pm x_i$ or $\pm \xi_4 x_i$. Thus the only choices for g are $\pm I$ and $\pm \xi_4$.

- Suppose \vec{x} is a 2-nilpotent tuple. Then the only other choices for the remaining x_j 's are $\pm \xi_{2^q}^k w$ or $\pm \xi_4 \xi_{2^q}^k w$. Hence g can only be $\pm I, \pm \xi_4, \pm \xi_{2^q}^k w$ or $\pm \xi_4 \xi_{2^q}^k w$.

We can conclude that if \vec{x} generates a copy of Q_8 , then $|\text{Stab}(\vec{x})| = 4$ and $|\text{Stab}(\vec{x})| = 2$ in any other case.

Corollary 2.12. *Let $q \geq 3$ and $n \geq 2$, then*

$$\text{Hom}(F_n/\Gamma_n^q, U(2)) \cong \text{Hom}(\mathbb{Z}^n, U(2)) \sqcup \bigsqcup_{\widetilde{M}(n,3)} (S^1)^n \times_{(\mathbb{Z}/2)^2} PU(2) \sqcup \bigsqcup_{\widetilde{M}(n,q)} (S^1)^n \times_{(\mathbb{Z}/2)} PU(2),$$

where $\widetilde{M}(n, q) = M(n, q - 1)$ is as in Corollary 2.11.

The obvious follow up question for these spaces is, for $m > 2$, what are the connected components of $\text{Hom}(F_n/\Gamma_n^q, U(m))$? We can not give an answer to this in its wide generality, but we can at least say something about the case $q = 3$.

2-nilpotent tuples in $U(m)$: Let \mathbf{a} be a partition of $\{1, 2, \dots, m\}$ into disjoint non-empty subsets. Define $U(\mathbf{a})$ as the subgroup of $U(m)$ consisting of $m \times m$ “block diagonal matrices with blocks indexed by \mathbf{a} ”, by which we mean matrices $A \in U(m)$ whose (i, j) -th entry is 0 whenever i and j are in different parts of the partition \mathbf{a} . To explain our terminology, notice that when each part of \mathbf{a} consists of consecutive numbers, say, if the

parts are $\{1, \dots, m_1\}$, $\{m_1 + 1, \dots, m_1 + m_2\}$, $\{m_1 + m_2 + 1, \dots, m_1 + m_2 + m_3\}$, \dots , then A is what is traditionally called a block diagonal matrix:

$$A = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_k \end{pmatrix}; \quad A_i \in U(m_i)$$

The conjugacy class of the subgroup $U(\mathbf{a})$ depends only on the sizes of the parts of \mathbf{a} . To be specific, if π is any permutation of $\{1, \dots, m\}$ such that the image of each part of \mathbf{a} consists of consecutive numbers, then $U(\mathbf{a})$ is conjugate, via the permutation matrix associated to π , to the subgroup of traditional block diagonal matrices as above (where the m_i are the sizes of the parts of \mathbf{a}). In particular, the subgroup $U(\mathbf{a})$ is always isomorphic to $\prod_{i=1}^k U(m_i)$ where the m_i are the sizes of the parts of \mathbf{a} but the isomorphism is not at all unique.

Let $Z_{\mathbf{a}}$ denote the center of $U(\mathbf{a})$. The elements of $Z_{\mathbf{a}}$ consist of “block scalar matrices”: diagonal matrices $\text{diag}(\lambda_1, \dots, \lambda_m) \in U(m)$ such that $\lambda_i = \lambda_j$ whenever i and j are in the same part of \mathbf{a} . For example, if the parts consist of consecutive numbers, the elements of $Z_{\mathbf{a}}$ are of the form:

$$\begin{pmatrix} \lambda_1 I_{m_1} & & 0 \\ & \ddots & \\ 0 & & \lambda_k I_{m_k} \end{pmatrix}.$$

Given any diagonal matrix $D = \text{diag}(\lambda_1, \dots, \lambda_m) \in U(m)$ there is a coarsest partition $\mathbf{a}(D)$ such that $D \in Z_{\mathbf{a}(D)}$, namely, the partition where i and j are in the same part *if and only if* $\lambda_i = \lambda_j$. One can easily check that the centralizer of D is precisely $U(\mathbf{a}(D))$.

Our goal is to interpret 2-nilpotent tuples of $U(m)$ as almost commuting elements of the subgroups $U(\mathbf{a})$, that is, for any topological group G and any closed subgroup $K \subset G$ contained in the center of G , the space of K -almost commuting n -tuples is the subspace of G^n where each (x_1, \dots, x_n) satisfies that $[x_i, x_j]$ lies in K for all i, j . We denote this space as $B_n(G, K)$ and we have an inclusion $B_n(G, Z(G)) \subset \text{Hom}(F_n/\Gamma_n^3, G)$.

In particular, for $U(m)$,

$$\bigcup_{\mathbf{a} \vdash m} B_n(U(\mathbf{a}), Z_{\mathbf{a}}) \subset \text{Hom}(F_n/\Gamma_n^3, U(m)),$$

where we’ve borrowed the notation $\mathbf{a} \vdash m$ typically used for partitions of the *number* m to indicate that the union is over all partitions of the *set* $\{1, \dots, m\}$ where each parts consists of consecutive numbers.

Let $\text{Rep}(\Gamma, G)$ denote the orbit space of the action of G on $\text{Hom}(\Gamma, G)$ by conjugation.

Proposition 2.13. *The above inclusion is surjective upon passing to orbits, that is,*

$$\left(\bigcup_{\mathbf{a} \vdash m} B_n(U(\mathbf{a}), Z_{\mathbf{a}}) \right) / U(m) = \text{Rep}(F_n/\Gamma_n^3, U(m)).$$

Remark 2.14. The notation $-/U(m)$ on the left is not meant suggest the union is closed under the conjugation action! It just means the image of the union in the orbit space.

Proof. Let (x_1, \dots, x_n) be an element of $\text{Hom}(F_n/\Gamma_n^3, U(m))$. Then every commutator $[x_i, x_j]$ is central in the group generated by $\{x_1, \dots, x_n\}$. In particular, all commutators commute with each other. Since each x_i is in $U(m)$ and hence diagonalizable, we can simultaneously diagonalize all commutators by an element $g \in U(m)$. Let $y_i := gx_i g^{-1}$ and $y_{ij} := g[x_i, x_j]g^{-1}$. Now, each y_{ij} lies in the center $Z_{\mathbf{a}_{ij}}$ for some coarsest partition \mathbf{a}_{ij} . Choose \mathbf{a} as the infimum of all the \mathbf{a}_{ij} , that is, as the coarsest partition refining all \mathbf{a}_{ij} . We have by construction $y_{ij} \in U(\mathbf{a}_{ij}) \subseteq U(\mathbf{a})$; and for each k , we have that y_k is in the centralizer of each y_{ij} , so $y_k \in \bigcap_k U(\mathbf{a}_{ij}) = U(\mathbf{a})$.

The last remaining detail is that this partition \mathbf{a} may not have parts that consist of consecutive number, but as explained above, a further conjugation fixes that. \square

Remark 2.15. The same argument works for $SU(m)$ and its subgroups $SU(\mathbf{a})$. There are only two minor differences: the first is that $SU(\mathbf{a})$ is identified with the subgroup of matrices in $\prod_{i=1}^k U(m_i)$ of determinant 1; the second is that for the last bit of the proof, the “consecutivization”, one needs to observe that for every partition there is an *even* permutation such that the image of each part consists of consecutive numbers and permutation matrices for even permutations lie in $SU(m)$.

3 STABLE HOMOTOPY TYPE

To compute the stable homotopy type of $\text{Hom}(F_n/\Gamma_n^q, SU(2))$ we need to describe $\text{Hom}(\mathbb{Z}^n, SU(2))$. This has been done independently in [2], [3] and [4].

Let $S_1(F_n/\Gamma_n^q, SU(2))$ denote the subspace of $\text{Hom}(F_n/\Gamma_n^q, SU(2))$ consisting of n -tuples with at least one coordinate equal to I . In [2] they show that

$$\text{Hom}(\mathbb{Z}^n, SU(2))/S_1(\mathbb{Z}^n, SU(2)) \cong \begin{cases} S^3 & \text{if } n = 1 \\ (\mathbb{RP}^2)^{n\lambda_2}/s_n(\mathbb{RP}^2) & \text{if } n \geq 2 \end{cases}$$

where $(\mathbb{RP})^{n\lambda_2}$ is the associated Thom space of $n\lambda_2$, n times the Whitney sum of the universal bundle λ_2 over \mathbb{RP}^2 , and s_n is its zero section. Using the homotopy stable decomposition of the simplicial space induced by the commuting tuples in a closed subgroup of $GL_n(\mathbb{C})$ (proved by the same authors in [1]) they get a complete description of $\text{Hom}(\mathbb{Z}^n, SU(2))$ after one suspension.

In [3], they prove that

$$\Sigma \text{Hom}(\mathbb{Z}^n, SU(2)) \simeq \Sigma \left(\bigvee_{k=1}^n \bigvee_{\binom{n}{k}} \Sigma S(k\lambda_2) \right)$$

where $S(k\lambda_2)$ is the sphere bundle associated to $k\lambda_2$. These two decompositions agree since $\Sigma S(k\lambda_2) \simeq (\mathbb{RP}^2)^{k\lambda_2}/s_n(\mathbb{RP}^2)$.

In M. C. Crabb’s paper [4], he expresses the stable homotopy type of $\text{Hom}(\mathbb{Z}^n, SU(2))_+$ as a wedge of various copies of \mathbb{RP}^2 , $\mathbb{RP}^4/\mathbb{RP}^2$ and $\mathbb{RP}^5/\mathbb{RP}^2$.

Now, it was proved in [7] that if G is a compact Lie group, then there are G -equivariant homotopy equivalences

$$\Sigma \text{Hom}(F_n/\Gamma_n^q, G) \simeq \bigvee_{1 \leq k \leq n} \Sigma \left(\bigvee^{\binom{n}{k}} \text{Hom}(F_k/\Gamma_n^q, G)/S_1(F_n/\Gamma_n^q, G) \right)$$

for all n and q . As a consequence of this and Theorem 2.10, we get the following:

Corollary 3.1. *Let $n \geq 1$. There are homotopy equivalences*

$$\Sigma \text{Hom}(F_n/\Gamma_n^q, SU(2)) \simeq \Sigma \bigvee_n S^3 \bigvee_{2 \leq k \leq n} \Sigma \left(\bigvee^{\binom{n}{k}} \left((\mathbb{RP}^2)^{k\lambda_2}/s_k(\mathbb{RP}^2) \vee \bigvee_{K(k,q)} \mathbb{RP}_+^3 \right) \right)$$

where

$$K(n, q) = \frac{7^n}{24} - \frac{3^n}{8} + \frac{1}{12} + \sum_{r=4}^q \frac{(2^r - 1)^n - 3(2^{r-1} - 1)^n + 2(2^{r-1} - 1)^n}{2^r}.$$

From Corollary 2.11 we get:

Corollary 3.2. *Let $n \geq 1$. Then $\Sigma \text{Hom}(F_n/\Gamma_n^q, SO(3))$ is homotopy equivalent to*

$$\Sigma \bigvee_n \mathbb{RP}^3 \bigvee_{2 \leq k \leq n} \Sigma \left((\mathbb{RP}^2)^{k\lambda_2}/s_k(\mathbb{RP}^2) \vee \left(\bigvee_{N(n)} (S^3/Q_8)_+ \right) \vee \bigvee^{\binom{n}{k}} \left(\bigvee_{N(k,q)} (S^3/C_4)_+ \right) \right)$$

where $N(n) = \frac{1}{2}(3^{n-1} - 1)$ and $N(n, q)$ is the solution to the recurrence relation

$$\sum_{k=1}^n N(k, q) \binom{n}{k} = M(n, q).$$

Now we describe the spaces of representations $\text{Rep}(F_n/\Gamma_n^q, G)$ of our groups of interest. In [2] they show that $\text{Rep}(\mathbb{Z}^n, SU(2))/((S_1(\mathbb{Z}^n, SU(2))/SU(2)) \cong S^n/\Sigma_2$ where the action of the generating element in Σ_2 is given by $(x_0, x_1, \dots, x_n) \mapsto (x_0, -x_1, \dots, -x_n)$ for any (x_0, \dots, x_n) in S^n . Identifying S^n with the suspension ΣS^{n-1} , we can see the above action as first taking antipodes on S^{n-1} and then suspending, that is $S^n/\Sigma_2 = \Sigma \mathbb{RP}^{n-1}$. Therefore

$$\Sigma \text{Rep}(F_n/\Gamma_n^q, SU(2)) \simeq \bigvee_{1 \leq k \leq n} \Sigma \left(\bigvee^{\binom{n}{k}} \left(\bigvee_{K(k,q)} S^0 \vee \Sigma \mathbb{RP}^{k-1} \right) \right).$$

Similarly

$$\Sigma \text{Rep}(F_n/\Gamma_n^q, SO(3)) \simeq \bigvee_{1 \leq k \leq n} \Sigma \left(\bigvee^{\binom{n}{k}} \left(\bigvee_{C(n)+N(k,q)} S^0 \vee \Sigma \mathbb{RP}^{k-1} \right) \right).$$

4 HOMOTOPY TYPE OF $B(r, Q_{2^q})$ FOR $r = 2, 3$ AND $B(q, SU(2))$

Now we turn our attention to the classifying spaces of transitionally q -nilpotent G -bundles for a topological group G . As described in [1], the spaces $\text{Hom}(F_n/\Gamma_n^q, G) \subset G^n$ give rise to a simplicial subspace of the nerve of G . The geometric realizations $B(q, G) := |\text{Hom}(F_*/\Gamma_*^q, G)|$ fit into a natural filtration of BG

$$B_{\text{com}}G := B(2, G) \subset B(3, G) \subset \cdots \subset B(q, G) \subset \cdots \subset BG.$$

So far we have completely described the subgroups of Q_{2^q} . This will allow us to compute the homotopy type of $B(r, Q_{2^q})$ as follows. Let G be a finite group. Consider the category $\mathcal{P}_r(G)$ with set of objects $\{M_\alpha, M_\alpha \cap M_\beta\}$ where M_α are the maximal subgroups of G of nilpotency class r . The set of arrows in $\mathcal{P}_r(G)$ is given by identities and inclusions. It was proved in [1] that when $\mathcal{P}_r(G)$ is a tree, there is a homotopy equivalence

$$B(r, G) \simeq B\left(\text{colim}_{A \in \mathcal{N}_r(G)} A\right).$$

Let $q \geq 4$. By Lemma 2.7 we conclude for $2 < r < q$

$$B(r, Q_{2^q}) \simeq B\left(*_{\mathbb{Z}/2^{r-1}} Q_{2^r} * *_{\mathbb{Z}/2} \mathbb{Z}/4\right)$$

where $*$ denotes the amalgamated product of groups.

Cohomology of $B_{\text{com}}Q_{2^q}$: Taking $r = 2$ we get that $B_{\text{com}}Q_{2^q} \simeq B(\mathbb{Z}/2^{q-1} *_{\mathbb{Z}/2} (*_{\mathbb{Z}/2} \mathbb{Z}/4))$ for any $q \geq 3$. Applying the associated Mayer-Vietoris sequence inductively we obtain

$$H^n(B_{\text{com}}Q_{2^q}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}/2^{q-1} \oplus (\mathbb{Z}/2)^{2^{q-2}} & n \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 4.1. $H^*(B_{\text{com}}Q_8; \mathbb{F}_2) \cong \mathbb{F}_2[y_1, y_2, y_3, z]/(y_i y_j, y_1^2 + y_2^2 + y_3^2, i \neq j)$ where y_i has degree 1 and z degree 2. Let $q \geq 4$. Then

$$H^*(B_{\text{com}}Q_{2^q}; \mathbb{F}_2) \cong \mathbb{F}_2[x_1, x_2, y_1, \dots, y_{2^{q-2}}, z]/(x_1^2, x_k y_i, y_i y_j, i \neq j, x_2 + \sum_{i=1}^{2^{q-2}} y_i^2),$$

where x_i, y_j have degree 1 and z degree 2.

Proof. We work out the case $q \geq 4$. Let $\Gamma = \mathbb{Z}/2^{q-1} *_{\mathbb{Z}/2} (*_{\mathbb{Z}/2} \mathbb{Z}/4)$. We use the central extension

$$\mathbb{Z}/2 \triangleleft \Gamma \rightarrow \mathbb{Z}/2^{q-2} * *_{\mathbb{Z}/2} \mathbb{Z}/2.$$

Recall that the cohomology rings for $n > 1$, $H^*(\mathbb{Z}/2^n; \mathbb{F}_2) \cong \mathbb{F}_2[x_1, x_2]/x_1^2$ where $\deg(x_1) = 1$ and $\deg(x_2) = 2$. Thus, $H^*(\mathbb{Z}/2^{q-2} * *_{\mathbb{Z}/2} \mathbb{Z}/2; \mathbb{F}_2) \cong \mathbb{F}_2[x_1, x_2, y_1, \dots, y_{2^{q-2}}]/(x_1^2, x_k y_i, y_i y_j, i \neq j)$. The k -invariant of the associated Serre-spectral sequence of this extension is $x_2 + \sum_{i=1}^{2^{q-2}} y_i^2$. Therefore the $E_3^{*,*}$ page is

$$\mathbb{F}_2[z] \otimes \mathbb{F}_2[x_1, x_2, y_1, \dots, y_{2^{q-2}}]/(x_1^2, x_k y_i, y_i y_j, i \neq j, x_2 + \sum_{i=1}^{2^{q-2}} y_i^2).$$

The Steenrod square $Sq^1(x_2 + \sum_{i=1}^{2^{q-2}} y_i^2) = Sq^1(x_2)$ is 0 since it can be expressed only in terms of x_2 . That is, $d_3 = 0$ and thus the spectral sequence abuts to $E_3^{*,*}$. \square

Cohomology of $B(3, Q_{16})$: We know that $B(3, Q_{16}) \cong B(*_{\mathbb{Z}/4}^2 Q_8 *_{\mathbb{Z}/2} *_{\mathbb{Z}/2}^4 \mathbb{Z}/4)$. To compute the \mathbb{F}_2 cohomology of this space, first we analyze $G_1 = Q_8 *_{\mathbb{Z}/4} Q_8$ by looking at the central extension

$$\mathbb{Z}/2 \triangleleft G_1 \rightarrow \mathbb{Z}/2 \times (\mathbb{Z}/2 * \mathbb{Z}/2)$$

and its k -invariant. Recall that $H^*(\mathbb{Z}/2 \times (\mathbb{Z}/2 * \mathbb{Z}/2); \mathbb{F}_2) \cong \mathbb{F}_2[x_1, x_2, z]/(x_1 x_2)$ with all generators of degree 1. By naturality, the k -invariant of this extension restricts to

$$\begin{array}{ccc} \mathbb{Z}/2 & \xlongequal{\quad} & \mathbb{Z}/2 \\ \downarrow & & \downarrow \\ Q_8 & \xrightarrow{\quad l \quad} & G_1 \\ \downarrow & & \downarrow \\ \mathbb{Z}/2 \times \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2 \times (\mathbb{Z}/2 * \mathbb{Z}/2) \end{array}$$

where l stands for the inclusion of one of the copies of Q_8 in G_1 . The k -invariant of the left extension in the diagram is $x^2 + xy + y^2$, where x, y are the degree 1 generators in $H^*((\mathbb{Z}/2)^2; \mathbb{F}_2)$ (recall that the cohomology ring $H^*(Q_8; \mathbb{F}_2) = \mathbb{F}_2[x, y, t]/(x^2 + xy + y^2, x^2 y + xy^2)$ where t has degree 4). Since the k -invariant of the right extension restricts to the inclusion of both copies of Q_8 in G_1 we conclude that the k -invariant of the desired extension must be $x_1^2 + x_2^2 + z^2 + x_1 z + x_2 z$. Now, let $G_2 = *_{\mathbb{Z}/2}^4 \mathbb{Z}/4$. The central extension $\mathbb{Z}/2 \triangleleft G_2 \rightarrow *^4 \mathbb{Z}/2$ has k -invariant $y_1^2 + y_2^2 + y_3^2 + y_4^2$ where $H^>(*^4 \mathbb{Z}/2; \mathbb{F}_2) \cong \mathbb{F}_2[y_1, y_2, y_3, y_4]/(y_i y_j, i \neq j)$ with all y_i of degree 1.

Using these k -invariants we can compute the cohomology of $G_1 *_{\mathbb{Z}/2} G_2$. Consider the central extension

$$\mathbb{Z}/2 \triangleleft G_1 *_{\mathbb{Z}/2} G_2 \rightarrow \mathbb{Z}/2 \times (\mathbb{Z}/2 * \mathbb{Z}/2) * *^4 \mathbb{Z}/2.$$

Let A denote the cohomology ring $H^*(\mathbb{Z}/2 \times (\mathbb{Z}/2 * \mathbb{Z}/2) * *^4 \mathbb{Z}/2; \mathbb{F}_2)$ which using the same notation as before, is isomorphic to $\mathbb{F}_2[x_1, x_2, y_1, y_2, y_3, y_4, z]/(x_1 x_2, y_i y_j, x_k y_i, z y_i, i \neq j)$. Let l denote 1 or 2. With a similar argument as for G_1 , the commutative diagram

$$\begin{array}{ccc} \mathbb{Z}/2 & \xlongequal{\quad} & \mathbb{Z}/2 \\ \downarrow & & \downarrow \\ G_l & \xrightarrow{\quad l \quad} & G_1 *_{\mathbb{Z}/2} G_2 \\ \downarrow & & \downarrow \\ G_l / \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2 \times (\mathbb{Z}/2 * \mathbb{Z}/2) * *^4 \mathbb{Z}/2 \end{array}$$

implies that the k -invariant for the right central extension in the diagram must be

$$k = x_1^2 + x_2^2 + z^2 + x_1 z + x_2 z + y_1^2 + y_2^2 + y_3^2 + y_4^2.$$

Using the Kudo and Serre transgression theorems, the differential d_2 in the E_2 -page of the Lyndon-Hochschild-Serre spectral sequence maps the generator t in $H^*(\mathbb{Z}/2; \mathbb{F}_2) = \mathbb{F}_2[t]$ to k which can be shown is not a zero divisor in A . Thus, writing an element in the E_2 -page, $E_2^{*,*} \cong \mathbb{F}_2[t] \otimes A$, as $a = \sum a_i t^i$, where $a_i \in A$, we have that $d_2(a) = \sum i a_i t^{i-1} k$. Since k is not a zero divisor in A , the E_3 -page is $E_3^{*,*} \cong \mathbb{F}_2[t^2] \otimes A/(k)$ where t^2 has degree 2. Next,

$$d_3(t^2) = Sq^1(k) = x_1^2 z + x_1 z^2 + x_2^2 z + x_2 z^2$$

and its image in the E_3 -page is z^3 . Again, writing a in the E_3 page as $\sum a_i (t^2)^i$, with $a_i \in A/(k)$, we can conclude that $\ker d_3 / \text{im } d_3$ which is the E_4 -page is

$$E_4^{*,*} \cong A/(k, z^3)[t^4] \oplus t^2 \text{ann}_{A/(k)}(z^3)[t^4].$$

where the annihilator $\text{ann}_{A/(k)}(z^3) = (y_1, y_2, y_3, y_4)$. The remaining differentials that could be non-zero are given by

$$d_{2n+1}(t^{2n}) = (x_1^{2n} + x_2^{2n})z + (x_1 + x_2)z^{2n}$$

for $n \geq 2$. In the E_{2n+1} -page, z^{2n} is zero, that is, the differentials are just $(x_1^{2n} + x_2^{2n})z$. For $n = 2$ we have that $d_5(t^4) = (x_1^4 + x_2^4)z$ with annihilator $\text{ann}_{A/(k, z^3)}(d_5(t^4)) = (y_1, y_2, y_3, y_4, z^2)$ in $A/(k, z^3)$. Proceeding as before, the E_6 -page is

$$E_6^{*,*} \cong A/(k, z^3, (x_1^4 + x_2^4)z)[t^8] \oplus t^4 \text{ann}_{A/(k, z^3)}((x_1^4 + x_2^4)z)[t^8] \oplus t^2 \text{ann}_{A/(k)}(z^3)[t^4].$$

The higher differentials are now zero. This completely describes the cohomology of $B(3, Q_{16})$.

Homotopy type of $B(q, SU(2))$: We finish with a homotopy pushout description of the spaces $B(q, SU(2))$. Consider the map $PU(2) \times (Q_{2q})^n \rightarrow \text{Hom}(F_n/\Gamma_n^q, SU(2))$ where (g, x_1, \dots, x_n) is mapped to $(gx_1g^{-1}, \dots, gx_ng^{-1})$ with $q > 2$. It is well defined in the quotient

$$PU(2) \times_{N_q} (Q_{2q})^n \rightarrow \text{Hom}(F_n/\Gamma_n^q, SU(2))$$

where the normalizer $N_q := N_{PU(2)}(D_{2q-1})$ acts by translation on $PU(2)$ and by conjugation on $(Q_{2q})^n$. Consider the subsets $\text{Gen}(n, Q_{2q}), \text{Hom}(F_n/\Gamma_n^{q-1}, Q_{2q}) \subset (Q_{2q})^n$ and the restrictions of the above map to the respective subspaces

$$\begin{array}{ccccc} \emptyset & \longrightarrow & PU(2) \times_{N_q} \text{Hom}(F_n/\Gamma_n^{q-1}, Q_{2q}) & \longrightarrow & \text{Hom}(F_n/\Gamma_n^{q-1}, SU(2)) \\ \downarrow & & \downarrow & & \downarrow \\ PU(2) \times_{N_q} \text{Gen}(n, Q_{2q}) & \hookrightarrow & PU(2) \times_{N_q} (Q_{2q})^n & \longrightarrow & \text{Hom}(F_n/\Gamma_n^q, SU(2)). \end{array}$$

We claim that all the above squares are pushouts. From the proof of Theorem 2.10 we have the homeomorphism

$$PU(2) \times \text{Gen}(n, Q_{2q})/N_q \cong \text{Hom}(F_n/\Gamma_n^q, SU(2)) - \text{Hom}(F_n/\Gamma_n^{q-1}, SU(2)).$$

Since $\text{Gen}(n, Q_{2q})$ is discrete, N_q is finite and acts freely, we have that

$$PU(2) \times \text{Gen}(n, Q_{2q})/N_q \cong PU(2) \times_{N_q} \text{Gen}(n, Q_{2q})$$

which proves the outside square to be a pushout in sets. But, $\text{Hom}(F_n/\Gamma_n^{q-1}, SU(2))$ is closed and $PU(2) \times_{N_q} \text{Gen}(n, Q_{2^q})$ is the image of a compact space, so that $\text{Hom}(F_n/\Gamma_n^q, SU(2))$ has the disjoint union topology. This proves our claim for the outside square, and a similar argument can be used for the square on the left side. Now we look at the right square. The outside and left squares being pushouts imply the right square is also a pushout. The middle arrow is a closed cofibration, and thus the right square is also a homotopy pushout. Moreover, since the maps are either inclusions or given by conjugation, all arrows are simplicial maps. Geometric realization commutes with colimits and homotopy colimits, and hence

$$\begin{array}{ccc} PU(2) \times_{N_q} B(q-1, Q_{2^q}) & \longrightarrow & B(q-1, SU(2)) \\ \downarrow & & \downarrow \\ PU(2) \times_{N_q} BQ_{2^q} & \longrightarrow & B(q, SU(2)) \end{array}$$

is a pushout of topological spaces and a homotopy pushout.

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